# Piecewise-Affine Approximation-Based Stochastic Optimal Control with Gaussian Joint Chance Constraints

Abraham P. Vinod*, Vignesh Sivaramakrishnan*, and Meeko M. K. Oishi

*Abstract*— **This paper considers the problem of stochastic optimal control of a Gaussian-perturbed linear system subject to soft polytopic state constraints, hard polytopic input constraints, and a convex cost function. We propose two conservative approaches using risk allocation that can be implemented via existing solvers, and characterize the approximations. Unlike existing approaches, we do not decouple the risk allocation from the optimal controller synthesis. We first show that risk allocation in conjunction with optimal controller synthesis introduces reverse convex constraints into the optimization problem. Next, we use piecewise-affine approximations of the nonlinear terms in the optimization problem to propose a mixed-integer convex program. Our piecewise-affine approximation produces a solver-friendly convex program when the safety probability threshold is larger than 0.5. Using two stochastic motion planning problems, we demonstrate that the proposed approach outperforms existing approaches like iterative risk allocation and particle control approaches in computation time, without compromising on the solution quality.**

## I. Introduction

Optimal control under uncertainty is an important problem, with applications ranging from path planning in robotics and space applications, to control of chemical plants [1]–[4]. A typical optimal control problem requires minimization of an objective, while ensuring that the system stays within a prescribed safe set and the control effort respects the hard actuator bounds. However, in real world applications, we must achieve optimal control while accounting for uncertainties like modeling errors, poor characterization of the environment, and limited sensing capabilities. While robust control approaches provide absolute guarantees of safety under uncertainty, this can often lead to conservative solutions, or even infeasibility, depending on the uncertainty. Given a stochastic characterization of the uncertainty, we can replace hard state constraints with soft constraints or chance constraints. Chance constraints permit the violation of the safety constraints with a probability below a specified threshold. By varying this threshold, we trade-off the conservativeness of the optimal trajectory, due to the safety constraints, with the cost associated with executing the trajectory. Reviews of robust and stochastic optimal control techniques in a model predictive framework are provided in [5]–[7].

In this paper, we will focus on the problem of stochastic optimal control of a Gaussian-perturbed linear system, subject to soft polytopic state constraints (*i.e.*, the probability of lying in a polytopic safe set must be above a threshold), hard polytopic input constraints (*i.e.*, design controllers must respect linear actuator bounds), and a convex cost function (*i.e.*, minimize actuation efforts or distance to a desired trajectory). The probabilistic safety constraints typically lead to a joint chance constraint that requires high dimensional integration and is hard to compute and enforce.

Among the several approaches that exist to solve the constrained stochastic optimal control problem [6], [7], we discuss the sampling-based [1], [8] and risk allocation-based approaches [9]–[11], since they can the handle the problem of interest without introducing severe conservatism. Sampling-based approaches approximate the uncertainty distribution using a finite number of particles, and formulate a mixed-integer optimization problem [1]. The main advantage of this approach is that it is distribution-independent, and has well-characterized lower bounds on the number of particles needed to achieve high quality solutions [8]. However, these lower bounds typically prescribe a large number of particles, resulting in computationally expensive or conservative optimization problems.

Another approach to handle joint chance constraints is risk allocation [9]–[11]. Here, the joint chance constraints are broken into individual chance constraints using Boole's inequality, and the violation probability threshold is broken into separate violation probability thresholds for each individual chance constraints. This converts the optimal control problem into an optimization problem for both optimal controller synthesis and risk allocation. For a safety probability threshold above $0.5$ (violation probability threshold below $0.5$), this problem is convex for open-loop controller synthesis [10] and for closed-loop controller synthesis under fixed risk allocation [9]. However, the individual chance constraints require enforcing constraints involving the cumulative density function or its inverse, the quantile function, and these constraints do not have a known conic reformulation for implementation in standard solvers [12], [13]. On the other hand, for a fixed risk allocation, the quantile function-based reformulation yields linear (for open-loop controller synthesis [10]) or second-order cone constraints (for closed-loop linear feedback controller synthesis [9], [11]). Existing approaches utilize this observation by decoupling the risk

This material is based upon work supported by the National Science Foundation under NSF Grant Number CMMI-1254990 and IIS-1528047. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

A. Vinod is with the Department of Aerospace Engineering and Engineering Mechanics, University of Texas at Austin, TX, US. This work was completed, while Vinod was a doctoral student at the University of New Mexico. Email: aby.vinod@gmail.com.

V. Sivaramakrishnan and M. Oishi are with Electrical & Computer Eng., University of New Mexico, Albq., NM. Email: {vigsiv,oishi}@unm.edu.

* These authors contributed equally to this work.

allocation problem from the optimal controller synthesis problem. The authors in [10], [11] propose iterative approaches where they tackle the risk allocation problem with various heuristics, while the authors in [9] use only fixed risk allocations. Due to the decoupled approach, none of these approaches can provide guarantees of optimality. Alternatively, researchers have proposed convex restrictions of chance-constrained optimization problems using conditional value-at-risk [14], whose implementation is typically done via sampling-based approximations [15].

We propose two iteration-free approaches in which the solver allocates the risk as well as synthesizes an open-loop controller. Our approaches are implementable in standard solvers and have simple user-defined parameters whose influence on the solution quality and computation time is explicit. By approximating nonlinear terms in the optimization problem as piecewise-affine constraints, we formulate a mixed-integer convex program without imposing any restrictions on the safety probability threshold. This formulation builds on the observation that the risk allocation in conjunction with optimal controller synthesis produces reverse convex constraints (constraints whose complement sets are convex). When the safety probability threshold is above $0.5$, we can use piecewise-affine approximations to formulate a convex program. We compare the efficacy of our approach to existing approaches [1], [10] on two stochastic motion planning problems: a robot with stochastic double integrator dynamics navigating a constrained environment, and the spacecraft rendezvous problem.

The main contributions of this paper are: 1) reformulation of the stochastic optimal control problem with risk allocation and optimal controller synthesis into an optimization problem with convex and reverse convex constraints, 2) conservative approximation (a mixed-integer convex program) of this reformulation using piecewise affine approximations when no restrictions are placed on the safety probability threshold, and 3) conservative and tractable approximation (a convex program) of the stochastic optimal control problem when the safety probability threshold is above $0.5$. For quadratic objectives, these problem formulations yield a mixed-integer quadratic program (MIQP) and a quadratic program (QP) respectively. We also study the effect of the choice of piecewise-affine approximation on the optimal controller synthesis as well as the risk allocation.

## II. Preliminaries and problem formulation

We denote a discrete-time time interval which inclusively enumerates all integers between $a$ and $b$ for $a, b \in \mathbb{N}$, $a \leq b$ by $\mathbb{N}_{[a,b]}$, random vectors with bold case, and non-random vectors with an overline. The indicator function of a non-empty set $\mathcal{E}$ is denoted by $1_{\mathcal{E}}(\overline{y})$, such that $1_{\mathcal{E}}(\overline{y}) = 1$ if $\overline{y} \in \mathcal{E}$ and is zero otherwise. We define a $p$-dimensional identity matrix as $I_p$, and we denote the Kronecker product with $\otimes$. We define the set $\mathcal{G}^N$ as the Cartesian product of a set $\mathcal{G} \subseteq \mathbb{R}^n$ with itself $N \in \mathbb{N}$ times.

### A. Reverse convex constraints

Reverse convex constraints arise when a convex function of the decision variable must be kept above a threshold. For example, in obstacle avoidance [1], [3], [11], we require the distance between a robot and an obstacle (a convex function) be above a threshold. That is, for some convex function $g : \mathbb{R}^n \to \mathbb{R}$, the constraint $\overline{x} \in \mathcal{E} = \{y : g(y) \geq 0\}$ must be maintained [16, Sec. 4.3.1]. This constraint is "reverse" convex because the complement of $\mathcal{E}$, the set $\mathbb{R}^n \setminus \mathcal{E} = \{y : g(y) < 0\}$, is convex. For a concave $h$, the constraint of the form $\overline{x} \in \{y : h(y) \leq 0\}$ results in a reverse convex constraint. Since $\mathcal{E}$ is non-convex, optimization problems involving reverse convex constraints are typically non-convex.

### B. Stochastic optimal control problem

We consider a stochastic linear time-varying system

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)\overline{u}(k) + \boldsymbol{w}(k) \tag{1}$$

with state $\boldsymbol{x}(k) \in \mathbb{R}^n$, input $\overline{u}(k) \in \mathcal{U} \subset \mathbb{R}^m$, and independent but not necessarily identical Gaussian process $\boldsymbol{w}(k) \in \mathbb{R}^p$, $\boldsymbol{w}(k) \sim \mathcal{N}(\overline{\mu}_{\boldsymbol{w}}(k), C_{\boldsymbol{w}}(k))$ with $\overline{\mu}_{\boldsymbol{w}}(k) \in \mathbb{R}^p$ and symmetric positive semidefinite $C_{\boldsymbol{w}}(k) \in \mathbb{R}^{p \times p}$ for $k \in \mathbb{N}$. We assume the initial state $\boldsymbol{x}(0) \in \mathbb{R}^n$ is Gaussian, $\boldsymbol{x}(0) \sim \mathcal{N}(\overline{\mu}_{\boldsymbol{x}}, C_{\boldsymbol{x}})$, with $\overline{\mu}_{\boldsymbol{x}} \in \mathbb{R}^n$ and symmetric positive semidefinite $C_{\boldsymbol{x}} \in \mathbb{R}^{n \times n}$. We assume a finite time horizon $N \in \mathbb{N}, N > 0$, $\boldsymbol{w}(k)$ and $\boldsymbol{x}(0)$ to be mutually independent for $k \in \mathbb{N}_{[0,N-1]}$, and a convex and compact $\mathcal{U}$.

We define the concatenated state vector, concatenated (deterministic) input vector, and concatenated disturbance vector as follows:

$$\boldsymbol{X} = \left[\boldsymbol{x}(0)^\top \ \ldots \ \boldsymbol{x}(N)^\top\right]^\top \in \mathbb{R}^{n(N+1)} \tag{2a}$$

$$\overline{U} = \left[\overline{u}(0)^\top \ \ldots \ \overline{u}(N-1)^\top\right]^\top \in \mathcal{U}^N \subset \mathbb{R}^{mN} \tag{2b}$$

$$\boldsymbol{W} = \left[\boldsymbol{w}(0)^\top \ \ldots \ \boldsymbol{w}(N-1)^\top\right]^\top \in \mathbb{R}^{pN} \tag{2c}$$

For a Gaussian process $\boldsymbol{w}(k)$, we have $\boldsymbol{W} \sim \mathcal{N}(\overline{\mu}_{\boldsymbol{W}}, C_{\boldsymbol{W}})$ with $\overline{\mu}_{\boldsymbol{W}} = \left[\overline{\mu}_{\boldsymbol{w}}(0)^\top \ \overline{\mu}_{\boldsymbol{w}}(1)^\top \ \ldots \ \overline{\mu}_{\boldsymbol{w}}(N-1)^\top\right]^\top \in \mathbb{R}^{pN}$ and $C_{\boldsymbol{W}} = \mathrm{blkdiag}(C_{\boldsymbol{w}}(0), \ldots, C_{\boldsymbol{w}}(N-1)) \in \mathbb{R}^{pN \times pN}$. From (1) and (2), we have

$$\boldsymbol{X} = \bar{A}\boldsymbol{x}(0) + H\overline{U} + G\boldsymbol{W} \tag{3}$$

where the matrices $\bar{A} \in \mathbb{R}^{n(N+1) \times n}$, $H \in \mathbb{R}^{n(N+1) \times mN}$, and $G \in \mathbb{R}^{n(N+1) \times pN}$ are obtained from the dynamics (1) (see [17] for their definitions). Recall that affine transformations of Gaussian random vectors are Gaussian [18, Ch. 3]. For a matrix $\Gamma \in \mathbb{R}^{n_y \times (pN)}$ ($n_y \in \mathbb{N}$) and vector $\overline{\nu} \in \mathbb{R}^{n_y}$,

$$\boldsymbol{W} \sim \mathcal{N}(\overline{\mu}_{\boldsymbol{W}}, C_{\boldsymbol{W}}) \xrightarrow{\boldsymbol{Y} = \Gamma\boldsymbol{W} + \overline{\nu}} \boldsymbol{Y} \sim \mathcal{N}(\overline{\mu}_{\boldsymbol{Y}}, C_{\boldsymbol{Y}}) \tag{4}$$

with $\overline{\mu}_Y = \Gamma\overline{\mu}_{\boldsymbol{W}} + \overline{\nu} \in \mathbb{R}^{n_y}$ and $C_{\boldsymbol{Y}} = \Gamma C_{\boldsymbol{W}} \Gamma^\top \in \mathbb{R}^{n_y \times n_y}$. Thus, for any $\overline{U} \in \mathbb{R}^{mN}$, we have $\boldsymbol{X}_{\overline{U}} \sim \mathcal{N}(\overline{\mu}_{\boldsymbol{X},\overline{U}}, C_{\boldsymbol{X},\overline{U}})$,

$$\overline{\mu}_{\boldsymbol{X},\overline{U}} = \bar{A}\overline{\mu}_{\boldsymbol{x}} + H\overline{U} + G\overline{\mu}_{\boldsymbol{W}}, \tag{5a}$$

$$C_{\boldsymbol{X},\overline{U}} = \bar{A}C_{\boldsymbol{x}}\bar{A}^\top + GC_{\boldsymbol{W}}G^\top. \tag{5b}$$

We are interested in solving the following stochastic optimal control problem,

$$\underset{\overline{U}\in\mathcal{U}^N}{\text{minimize}} \quad \mathbb{E}_{\boldsymbol{X}}^{\overline{U}}\left[J(\boldsymbol{X}_{\overline{U}},\overline{U})\right] \tag{6a}$$
$$\text{subject to} \quad \mathbb{P}_{\boldsymbol{X}}^{\overline{U}}\left\{\boldsymbol{X}_{\overline{U}}\in\mathscr{S}\right\}\geq 1-\Delta \tag{6b}$$

with decision variable $\overline{U}\in\mathbb{R}^{mN}$, convex cost function $J:\mathbb{R}^{n(N+1)}\times\mathcal{U}^N\to\mathbb{R}$, a polytopic set of hard constraints on the input $\mathcal{U}^N\subset\mathbb{R}^{mN}$, a polytopic set $\mathscr{S}=\{\overline{X}\in\mathbb{R}^{n(N+1)}: P\overline{X}\leq\overline{q}\}\subseteq\mathbb{R}^{n(N+1)}$ in which the state should lie, and a lower bound on its probability of satisfaction, $1-\Delta\in(0,1]$. Thus, the constraint (6b) lower bounds the safety probability $\mathbb{P}_{\boldsymbol{X}}^{\overline{U}}\left\{\boldsymbol{X}_{\overline{U}}\in\mathscr{S}\right\}$ associated with the controller $\overline{U}$. In defining $\mathscr{S}$, we have $P=\left[\overline{p}_1^\top\ \ldots\ \overline{p}_{L_X}^\top\right]^\top\in\mathbb{R}^{L_X\times n(N+1)}$ and $\overline{q}=\left[q_1\ \ldots\ q_{L_X}\right]^\top\in\mathbb{R}^{L_X}$ with $L_X\in\mathbb{N}$.

As in typical optimal control problems, we consider a quadratic cost function $J(\boldsymbol{X}_{\overline{U}},\overline{U})=\overline{U}^\top R\overline{U}+(\boldsymbol{X}_{\overline{U}}-\overline{X}_{\mathrm{d}})^\top Q(\boldsymbol{X}_{\overline{U}}-\overline{X}_{\mathrm{d}})$ for some symmetric positive semi-definite matrices $Q\in\mathbb{R}^{n(N+1)\times n(N+1)}$ and $R\in\mathbb{R}^{(mN)\times(mN)}$ to penalize the control effort and the distance from some desired trajectory $\overline{X}_{\mathrm{d}}\in\mathscr{S}$. For Gaussian $\boldsymbol{X}_{\overline{U}}$ (5) and quadratic cost $J(\cdot)$, (6a) has a closed-form expression which is quadratic in $\overline{U}$ [18, Sec. 7.5],

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{X}}^{\overline{U}}\left[J(\boldsymbol{X}_{\overline{U}},\overline{U})\right]\\&=\overline{U}^\top R\overline{U}+(\overline{\mu}_{\boldsymbol{X},\overline{U}}-\overline{X}_{\mathrm{d}})^\top Q(\overline{\mu}_{\boldsymbol{X},\overline{U}}-\overline{X}_{\mathrm{d}})+\mathrm{tr}(QC_{\boldsymbol{X},\overline{U}}).\end{aligned} \tag{7}$$

Here, tr denotes the trace operator. By (5b), $\mathrm{tr}(QC_{\boldsymbol{X},\overline{U}})$ is independent of the decision variable $\overline{U}$. Clearly, $\mathbb{E}_{\boldsymbol{X}}^{\overline{U}}\left[J(\boldsymbol{X}_{\overline{U}},\overline{U})\right]$ is convex in $\overline{U}$ by the positive semi-definiteness of $Q$ and $R$, and (5a) [19, Ex. 3.49].

### C. Risk allocation approach

Even with a convex quadratic cost (7), the problem (6) is hard to solve due to the *joint chance constraint* (6b). Risk allocation transforms the joint chance constraints (6b) to a set of *individual chance constraints* via Boole's inequality, posed as linear constraints or second-order cone constraints using the properties of Gaussian random vectors [4], [9]–[11]. Specifically, we tighten (6b) to obtain the following set of tractable constraints,

$$\Delta\geq\sum_{i=1}^{L_X}\delta_i,\quad \delta_i\geq 0,\qquad \forall i\in\mathbb{N}_{[1,L_X]} \tag{8a}$$
$$\Phi\left(\frac{q_i-\overline{p}_i^\top\overline{\mu}_{\boldsymbol{X},\overline{U}}}{\|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2}\right)\geq 1-\delta_i,\qquad \forall i\in\mathbb{N}_{[1,L_X]} \tag{8b}$$

with $\Phi(\cdot)$ as the standard normal cumulative density function (CDF). Any controller $\overline{U}\in\mathcal{U}^N$ that satisfies (8) along with a risk allocation $\overline{\delta}\triangleq[\delta_1\ \ldots\ \delta_{L_X}]\in[0,1]^{L_X}$, satisfies (6b).

Using (8), the authors in [9], [10] have proposed the following *restriction* of (6),

$$\underset{\overline{U},\overline{\delta},\overline{\mu}_{\boldsymbol{X},\overline{U}}}{\text{minimize}} \quad \mathbb{E}_{\boldsymbol{X}}^{\overline{U}}[J(\boldsymbol{X}_{\overline{U}},\overline{U})] \tag{9a}$$
$$\text{subject to} \quad \Delta\geq\sum_{i=1}^{L_X}\delta_i,\quad \text{(5a)},\quad \overline{U}\in\mathcal{U}^N \tag{9b}$$
$$\delta_i\geq 0,\qquad \forall i\in\mathbb{N}_{[1,L_X]} \tag{9c}$$
$$\overline{p}_i^\top\overline{\mu}_{\boldsymbol{X},\overline{U}}+\|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2\Phi^{-1}(1-\delta_i)\leq q_i,\quad \forall i\in\mathbb{N}_{[1,L_X]} \tag{9d}$$

Recall that $\Phi^{-1}(\cdot)$ is a monotone function. The constraint (9d) is equivalent to the constraint (8b), as it follows from applying $\Phi^{-1}(\cdot)$ to both sides of the inequality (8b), and rearranging the terms.

For $\Delta\leq 0.5$, (9) is a *convex* restriction [10, Thm. 1]. Since $\Phi^{-1}(1-\delta_i)$ does not have a conic reformulation and (9d) is linear under fixed-risk allocation, the authors in [10] proposed an iterative approach that solves (9). They solve simpler convex programs with fixed risk allocation $\overline{\delta}$, update the risk allocations, and repeat these steps till convergence. Here, we propose tractable restrictions of the optimal control problem (9) such that the solver can handle risk allocation as well as the optimal controller synthesis simultaneously, without imposing the requirement that $\Delta\leq 0.5$.

### D. Problem statements

**Problem 1.** *Obtain a mixed-integer convex program restriction of* (9) *for any* $\Delta\in[0,1)$.

**Problem 1a.** *Show that* (9) *is equivalent to an optimization problem with convex and reverse convex constraints, and a convex objective for any* $\Delta\in[0,1)$.

We exploit the structure in the reverse convex constraints to tractably solve (9) via mixed-integer convex programming.

**Problem 1b.** *Use piecewise-affine approximation and the "big-M" approach to tighten the reverse convex constraints using mixed-integer affine constraints.*

**Problem 2.** *For* $\Delta\in[0,0.5)$*, obtain a convex restriction of* (9) *using a piecewise-affine overapproximation of* $\Phi^{-1}(1-\delta_i)$.

## III. Mixed-integer convex program for $\Delta\in[0,1)$

### A. Risk allocation yields reverse convex constraints

Consider the following optimization problem,

$$\underset{\overline{U},\overline{\delta},\overline{s},\overline{\mu}_{\boldsymbol{X},\overline{U}}}{\text{minimize}} \quad \mathbb{E}_{\boldsymbol{X}}^{\overline{U}}\left[J(\boldsymbol{X}_{\overline{U}},\overline{U})\right] \tag{10a}$$
$$\text{subject to} \quad \Delta\geq\sum_{i=1}^{L_X}\delta_i,\quad \text{(5a)},\quad \overline{U}\in\mathcal{U}^N \tag{10b}$$
$$\delta_i\in[0,\Delta],\ \ s_i\in[\log(1-\Delta),0],\forall i\in\mathbb{N}_{[1,L_X]} \tag{10c}$$
$$\log\Phi\left(\frac{q_i-\overline{p}_i^\top\overline{\mu}_{\boldsymbol{X},\overline{U}}}{\|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2}\right)\geq s_i,\quad \forall i\in\mathbb{N}_{[1,L_X]} \tag{10d}$$
$$\log(1-\delta_i)\leq s_i,\qquad \forall i\in\mathbb{N}_{[1,L_X]} \tag{10e}$$

with $\overline{s} = [s_1\ \dots\ s_{L_X}] \in \mathbb{R}^{L_X}$ as a vector of slack variables.

**Proposition 1.** (10) *is equivalent to* (9), $\forall \Delta \in [0,1)$.

*Proof:* The cost function (10a) and the constraint (10b) are identical to (9a) and (9b). We obtain (10d) and (10e) by reformulating (8b) using $\log(\cdot)$ and introducing the slack variables $s_i$ to enforce the inequality [19, Sec. 4.1.3]. By (8a), $\delta_i \in [0,\Delta]\ \forall i \in \mathbb{N}_{[1,L_X]}$. We have $s_i \in [\log(1-\Delta), 0]\ \forall i \in \mathbb{N}_{[1,L_X]}$ in (10c) due to the bounds on $s_i$ enforced by (10d) and (10e). Recall that $\log(\Phi(y)) \le 0\ \forall y \in \mathbb{R}$ and $\log(1-y)$ is monotone decreasing in $y$. ■

**Proposition 2.** *Problem* (10) *has a convex objective, and convex and reverse convex constraints.*

*Proof:* The cost (10a) and the constraints (10b) and (10c) are convex [19, Sec. 3.2]. We know that the standard normal CDF $\Phi(y)$ is log-concave over $y \in \mathbb{R}$, and $\log(1-y)$ is concave over $y < 1$ [19, Ch. 3]. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is log-concave if $f(y) \ge 0\ \forall y \in \mathbb{R}$ and $\log(f(y))$ is concave with $\log 0 \triangleq -\infty$ [19, Sec. 3.5.1]. Thus, (10d) is also convex. However, the constraint (10e) is reverse convex, since $\{\log(1-\delta_i) - s_i\}$ is concave in $(s_i, \delta_i)$ and (10e) defines the complement of a convex set. ■

Propositions 1 and 2 solve Problem 1a. Note that the constraints (10d) and (10e) are not easy to enforce, because the latter is non-convex and the former, though convex, does not have a known reformulation into conic constraints. Recall that standard convex solvers can handle only conic constraints [12], [13]. We will propose a tractable restriction of (10) using piecewise affine approximations and the presence of reverse convex constraints in (10).

### B. Tightening constraints by piecewise-affine approximation

Consider a concave function $f(x) : \mathcal{D} \to \mathcal{R}$, with bounded $\mathcal{D}, \mathcal{R} \subset \mathbb{R}$, convex $\mathcal{D} = [x_{\min}, x_{\max}]$, and $x_{\min}$, $x_{\max} \in \mathbb{R}$, $x_{\min} < x_{\max}$. We use piecewise-affine (PWA) approximations to tighten constraints of the form $f(x) \ge s$ (a convex constraint) and $f(x) \le s$ (a reverse convex constraint) for some $s \in \mathbb{R}$.

We construct PWA approximations of $f(x)$ as $\ell_f^+, \ell_f^- : \mathbb{R} \to \mathbb{R}$ for some $m_{f,j}^+, c_{f,j}^+, m_{f,j}^-, c_{f,j}^- \in \mathbb{R}\ \forall j \in \mathbb{N}_{[1,N_f]}$,

$$\ell_f^-(x) \triangleq \min_{j \in \mathbb{N}_{[1,N_f]}} \{m_{f,j}^- x + c_{f,j}^-\}, \tag{11a}$$

$$\ell_f^+(x) \triangleq \min_{j \in \mathbb{N}_{[1,N_f]}} \{m_{f,j}^+ x + c_{f,j}^+\} \tag{11b}$$

such that $\forall x \in \mathcal{D}$ and a given approximation error $\eta > 0$,

$$f(x) - \eta \le \ell_f^-(x) \le f(x) \le \ell_f^+(x) \le f(x) + \eta. \tag{12}$$

Due to the concavity of $f(x)$, the PWA underapproximation $\ell_f^-(x)$ can be constructed using the secants obtained by connecting $(x_i, f(x_i))$ for a collection of $x_i \in \mathcal{D}$, and the PWA overapproximation $\ell_f^+(x)$ can be constructed by shifting these secants to define tangents of $f(x)$ [19, Sec. 3.1]. Figure 1 illustrates this approximation using hypographs. Recall that the hypograph of a function $f(x)$ is the set $\{(x,s) : f(x) \ge s\}$ and its epigraph is the set $\{(x,s) : f(x) \le s\}$ [19, Sec. 3.1.7].

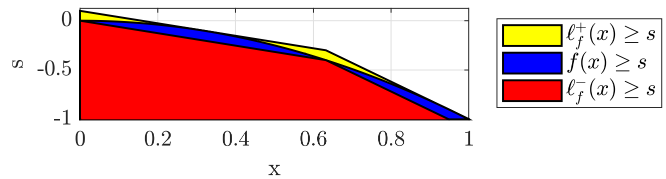


Fig. 1. Piecewise-affine approximation of $f(x) = -x^2$ over $\mathcal{D} = [0,1]$ with hypographs of $\ell_f^-(x), \ell_f^+(x), f(x)$ shaded.

The choice of $x_i$ (breakpoints) determines the accuracy of the PWA approximation. Appendix A discusses an algorithm (Algorithm 1) to choose these breakpoints based on the user-provided approximation tolerances. This approach satisfies (12) using linear Lagrange interpolation [20, Eq. 25.2.1].

By (12), the hypograph of $\ell_f^-(x)$ is a subset of the hypograph of $f(x)$, while the epigraph of $\ell_f^-(x)$ is a subset of the epigraph of $f(x)$,

$$\{(x,s) : \ell_f^-(x) \ge s\} \subseteq \{(x,s) : f(x) \ge s\}, \tag{13a}$$

$$\{(x,s) : \ell_f^+(x) \le s\} \subseteq \{(x,s) : f(x) \le s\}. \tag{13b}$$

By (11), we have

$$\ell_f^-(x) \ge s \Leftrightarrow \forall j \in \mathbb{N}_{[1,N_f]}, m_{f,j}^- x + c_{f,j}^- \ge s, \tag{14a}$$

$$\ell_f^+(x) \le s \Leftrightarrow \exists j \in \mathbb{N}_{[1,N_f]}, m_{f,j}^+ x + c_{f,j}^+ \le s. \tag{14b}$$

By (14a), we can impose the convex constraint of $f(x) \ge s$ using a collection of affine constraints. On the other hand, we can tighten the constraint $f(x) \le s$ (a reverse convex set in $(x,s)$) by enforcing an *optional* constraint satisfaction (14b). We utilize the "big-M" approach [21], a mixed-integer constraint reformulation to enforce $\ell_f^+(x) \le s$. For every $j \in \mathbb{N}_{[1,N_f]}$, we define a binary decision variable $\xi_j \in \{0,1\}$,

$$\{m_{f,j}^+ x + c_{f,j}^+ \le s\} \Leftrightarrow [\xi_j = 1]. \tag{15}$$

The bijection in (14b) may be equivalently enforced by the following (mixed-integer) affine constraints,

$$\begin{array}{rll} m_{f,j}^+ x + c_{f,j}^+ - s & \le M_j^{\mathrm{ub}}(1 - \xi_j), & \forall j \in \mathbb{N}_{[1,N_f]} \\ m_{f,j}^+ x + c_{f,j}^+ - s & \ge \epsilon + (M_j^{\mathrm{lb}} - \epsilon)\xi_j, & \forall j \in \mathbb{N}_{[1,N_f]} \\ \xi_j & \in \{0,1\}, & \forall j \in \mathbb{N}_{[1,N_f]} \\ \sum_{j=1}^{N_f} \xi_j & \ge 1, & \end{array} \tag{16}$$

with $\epsilon$ as a small tolerance (typically the machine precision), and $M_j^{\mathrm{ub}}$ and $M_j^{\mathrm{lb}}$ as the (constant) upper and lower bounds of $m_{f,j}^+ x + c_{f,j}^+ - s$ in the range of values of $(x,s)$. We compute $M_j^{\mathrm{ub}}$ and $M_j^{\mathrm{lb}}$ offline. The existence of $j \in \mathbb{N}_{[1,N_f]}$ for optional constraint satisfaction (14b) is enforced by requiring $\sum_{j=1}^{N_f} \xi_j \ge 1$, at least one of the binary decision variables $\xi_j$ is 1.

### C. Construction of the mixed-integer convex program

We now construct a restriction of the optimization problem (10), a mixed-integer convex program (17) (see the next page). In (17), the cost function (17a) and the constraint (17b) are identical to (10a) and (10b) respectively.

To tighten (10d), we analyze the function $h(y) = \log(\Phi(y))$, $y \in \mathbb{R}$. It is easy to check that $h(y)$ is a concave

$$\begin{aligned}
\underset{\overline{U},\overline{\delta},\overline{s},\overline{\mu}_{\boldsymbol{X},\overline{U}},\overline{\xi}}{\text{minimize}} \quad & \mathbb{E}_{\boldsymbol{X}}^{\overline{U}}\left[J(\boldsymbol{X}_{\overline{U}},\overline{U})\right] && \text{(17a)}\\
\text{subject to} \quad & \Delta \geq \sum\nolimits_{i=1}^{L_X}\delta_i,\quad \overline{\mu}_{\boldsymbol{X},\overline{U}} = \bar{A}\overline{\mu}_{\boldsymbol{x}} + H\overline{U} + G\overline{\mu}_{\boldsymbol{W}},\quad \overline{U}\in\mathcal{U}^N, && \text{(17b)}\\
& \delta_i\in[0,\Delta],\quad s_i\in[\log(1-\Delta),-\eta_h], \qquad \forall i\in\mathbb{N}_{[1,L_X]}, && \text{(17c)}\\
& m_{h,j}^{-}(q_i-\overline{p}_i^\top\overline{\mu}_{\boldsymbol{X},\overline{U}}) + c_{h,j}^{-}\|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2 \geq s_i\|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2, \qquad \forall i\in\mathbb{N}_{[1,L_X]},\forall j\in\mathbb{N}_{[1,N_h]}, && \text{(17d)}\\
& q_i-\overline{p}_i^\top\overline{\mu}_{\boldsymbol{X},\overline{U}} \geq -K\|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2, \qquad \forall i\in\mathbb{N}_{[1,L_X]}, && \text{(17e)}\\
& m_{g,j}^{+}\delta_i + c_{g,j}^{+} - s_i \leq M_j^{\mathrm{ub}}(1-\xi_{ij}), \qquad \forall i\in\mathbb{N}_{[1,L_X]},\forall j\in\mathbb{N}_{[1,N_g]}, && \text{(17f)}\\
& m_{g,j}^{+}\delta_i + c_{g,j}^{+} - s_i \geq \epsilon + (M_j^{\mathrm{lb}}-\epsilon)\xi_{ij}, \qquad \forall i\in\mathbb{N}_{[1,L_X]},\forall j\in\mathbb{N}_{[1,N_g]}, && \text{(17g)}\\
& \xi_{ij}\in\{0,1\}, \qquad \forall i\in\mathbb{N}_{[1,L_X]},\forall j\in\mathbb{N}_{[1,N_g]}, && \text{(17h)}\\
& \sum\nolimits_{j=1}^{N_g}\xi_{ij}\geq 1, \qquad \forall i\in\mathbb{N}_{[1,L_X]}. && \text{(17i)}
\end{aligned}$$

function with an unbounded domain and range. To construct a PWA approximation, we restrict the domain of $h(y)$ to $[-K,\Phi^{-1}(e^{-\eta_h})]$ for a positive real number $K$ and the user-specified approximation error bound $\eta_h > 0$. The upper bound is motivated by the fact $\log(\Phi(y)) \to 0$ as $y\to\infty$, and for every $y \geq \Phi^{-1}(e^{-\eta_h})$, $\log(\Phi(y)) \in [-\eta_h, 0)$. The lower bound $-K$ should ensure that $\Phi(-K)$ is sufficiently close to zero. For $y\in[-K,\infty)$, $\ell_h^-(y)$ underapproximates $h(y)$ within $\eta_h$ using $N_h+1$ affine segments in the domain $[-K,\infty)$,

$$\ell_h^-(y) = \min_{j\in\mathbb{N}_{[1,N_h]}}\{m_{h,j}^- y + c_{h,j}^-, -\eta_h\} \tag{18}$$

for an appropriate choice of $m_{h,j}^-, c_{h,j}^-$. The restriction in domain of $h(y)$ introduces the constraint (17e), and updates (10c) to (17c). Using (13a) and (14a), we enforce (10d) in (17) via (17c), (17d), and (17e).

To tighten the constraint (10e), we analyze the function $g(y) = \log(1-y),\ y\in[0,\Delta]$. It is easy to check that $g(y)$ is also a concave function with a bounded domain $[0,\Delta]$ and range $[\log(1-\Delta),0]$. Thus, we can construct $\ell_g^+(y)$, a PWA overapproximation to $g(y)$ with an error bound $\eta_g>0$,

$$\ell_g^+(y) \triangleq \min_{j\in\mathbb{N}_{[1,N_g]}}\{m_{g,j}^+ y + c_{g,j}^+\},\ \forall y\in[0,\Delta]. \tag{19}$$

We enforce the reverse convex constraint (10e) using (13b), (15), and (16), and a binary decision vector $\overline{\xi} = [\xi_{11}\ \ldots\ \xi_{1N_g}\ \xi_{21}\ \ldots\ \xi_{L_X N_g}]^\top \in \{0,1\}^{L_X N_g}$, to obtain the constraints (17f), (17g), (17h), and (17i). Note that $g(y)-s$ is monotone decreasing in $y$ and $s$. From (17c), the lower and upper bounds on $g(y)-s$ for (16) are

$$M_j^{\mathrm{lb}} = c_{g,j}^+ + m_{g,j}^+\Delta + \eta_h \qquad \forall j\in\mathbb{N}_{[1,N_g]}, \tag{20a}$$
$$M_j^{\mathrm{ub}} = c_{g,j}^+ - \log(1-\Delta) \qquad \forall j\in\mathbb{N}_{[1,N_g]}. \tag{20b}$$

**Proposition 3.** *For any $\Delta\in[0,1)$, (17) is a mixed-integer convex restriction of (9).*

*Proof:* Follows from Proposition 1, (13), and the fact that (17a) is convex and all constraints of (17) are affine. ■

Proposition 3 solves Problem 1b, and thereby Problem 1. Every feasible solution of (17) is feasible for (10) and (9).

**Remark 1.** *For a quadratic cost (7), (17) is a MIQP.*

### D. Effect of user-defined parameters

In (17), there are three user-defined parameters required to construct $\ell_g^+(y), \ell_h^-(y)$: maximum approximation errors $\eta_g,\eta_h>0$ and the domain restriction $K$ for $h(y)$. Using a small value of $\eta_g,\eta_h$ and/or large values of $K$ makes (17) a closer representation of (10). This, however, increases the number of affine segments in the PWA approximation, which in turn increases the computation cost due to a increase in the number of constraints (larger $N_g, N_h$) and more binary variables (in case of $\ell_g^+(y)$). On the other hand, a larger value of $\eta_g,\eta_h$ and/or reducing $K$ results in a fewer constraints and improves the computation time at the cost of additional conservativeness. Thus, we obtain a tradeoff between the computation time and accuracy (conservativeness) by the choice of $K$, $\eta_h$, and $\eta_g$.

In summary, given a stochastic optimal control problem (9), we exploited the log-concavity of the CDF to propose an equivalent reformulation in (10). Next, we enforced the nonlinear (convex and reverse convex) constraints using their PWA approximations to obtain the mixed-integer convex program (17). Finally, we showed that (17) enables a tractable solution to (9). We recommend the use of Algorithm 1 in Appendix A to construct the PWA approximations up to a user-specified tolerance.

## IV. Convex program for $\Delta \leq 0.5$

Recall that (9) is convex under the restriction of $\Delta\leq 0.5$. Since the constraint (9d) cannot be reformulated into a conic constraint, we use the PWA overapproximation of the convex function $f(y) = \Phi^{-1}(1-y),\ y\in[0,\Delta]$,

$$\ell_f^+(y) \triangleq \max_{j\in\mathbb{N}_{[1,N_f]}}\{m_{f,j}^+ y + c_{f,j}^+\},\ y\in[\delta_{\mathrm{lb}},\Delta]. \tag{21}$$

We restrict the domain of $f(y)$ to $y\in[\delta_{\mathrm{lb}},0.5]$ for a small $\delta_{\mathrm{lb}}\in\mathbb{R}, \delta_{\mathrm{lb}}>0$ since $\Phi^{-1}(1-y)\to\infty$ as $y\to 0^+$.

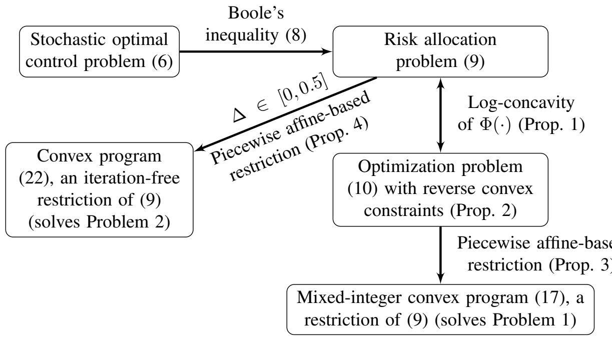


Fig. 2. Relationship between various optimization problems presented in this paper. While (9) and (10) are equivalent, all other relations are restrictions (one-directional relationships). For a quadratic cost (7), (17) and (22) are mixed-integer quadratic and quadratic programs respectively.

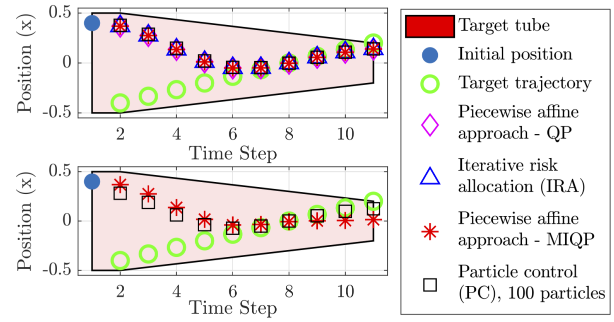


Fig. 3. Double integrator example with safety probability threshold $1-\Delta = 0.8$ (top plot) and $1-\Delta = 0.4$ (bottom plot).

To use Algorithm 1 of Appendix A, we compute the PWA underapproximation of $-f(y)$ (a concave function).

Using (21) and (13a), we have the following problem,

$$\underset{\overline{U},\overline{\delta},\overline{\mu}_{\boldsymbol{X},\overline{U}}}{\text{minimize}} \quad \mathbb{E}_{\boldsymbol{X}}^{\overline{U}}\left[J(\boldsymbol{X}_{\overline{U}},\overline{U})\right] \tag{22a}$$

$$\text{subject to} \quad \Delta \geq \sum\nolimits_{i=1}^{L_X} \delta_i, \quad \text{(5a)}, \quad \overline{U}\in\mathcal{U}^N, \tag{22b}$$

$$\delta_i \in [\delta_{\mathrm{lb}}, \Delta], \qquad \forall i\in\mathbb{N}_{[1,L_X]} \tag{22c}$$

$$\overline{p}_i^\top \overline{\mu}_{\boldsymbol{X},\overline{U}} + \|C_{\boldsymbol{X},\overline{U}}^{\frac{1}{2}}\overline{p}_i\|_2 \left(m_{f,j}^+\delta_i + c_{f,j}^+\right) \leq q_i, \quad \forall i\in\mathbb{N}_{[1,L_X]}, \forall j\in\mathbb{N}_{[1,N_f]} \tag{22d}$$

**Proposition 4.** *Problem* (22) *is a convex restriction of* (9) *when* $\Delta\in[0,0.5]$.

*Proof:* While (22a) and (22b) are identical to (9a) and (9b), (22c) and (22d) are restrictions of (9c) and (9d) respectively. We complete the proof by noting that (22) minimizes a convex cost over affine constraints. ■

**Remark 2.** *For a quadratic cost* (7), (22) *is a QP.*

Proposition 4 solves Problem 2. By construction, the convex solver allocates the risk and synthesizes an optimal controller in a *single* optimization problem (22). In contrast, the iterative risk allocation decouples the risk allocation and controller synthesis problem, and solves the optimal control problem in an iterative manner [10]. By Proposition 4, every feasible solution of (22) is feasible for (9).

This approach requires two user-defined parameters, $\delta_{\mathrm{lb}}$ and $\eta_f$. Similarly to Section III-D, large $\delta_{\mathrm{lb}}$ and/or large $\eta_f$ will reduce the number of affine segments, which improves the computation time but with additional conservativeness. We choose $\delta_{\mathrm{lb}} \ll \Delta/L_X$ to ensure that the solution to (22) is not solely determined by (22b) and (22c).

Figure 2 summarizes the relationships between the optimization problems introduced in this paper.

## V. Numerical Study

We implement our proposed solutions on two stochastic motion planning problems with quadratic cost objectives (7). We compare the QP to solve (22) with the iterative risk allocation approach (IRA) in [10] when $1-\Delta \geq 0.5$, and compare the MIQP to solve (17) with the particle control approach (PC) in [1] for arbitrary $1-\Delta\in(0,1]$. For the implementation of the PWA approximations, we chose $\eta_g = \eta_h = 5\times10^{-4}$ and $K=5$ for (17), $\eta_f = 10^{-2}$ and $\delta_{\mathrm{lb}} = 10^{-5}$ for (22). We also perform a Monte-Carlo simulation-based validation of the optimal controllers using $10^5$ particles. Lastly, we compare the relative absolute error in the simulated cost with the expected cost, and evaluate the simulated safety probability against the prescribed safety probability threshold. Since PC produces different controllers under each run, we report the average results from three executions of the method. The proposed MIQP approach is sampling-free and provides consistent results, a clear advantage over the PC method.

All computations were done using MATLAB on an Intel Xeon CPU with 3.80GHz clock rate and 32 GB RAM. We used CVX [12] with Gurobi [13] as the underlying solver to solve (17) and (22) and to implement IRA [10] and PC [1]. We used MPT [22] and SReachTools [23] for the stochastic optimal control problem formulation.

### A. Double integrator example

We consider a double integrator system,

$$\boldsymbol{x}(k+1) = \begin{bmatrix}1 & T_s\\ 0 & 1\end{bmatrix}\boldsymbol{x}(k) + \begin{bmatrix}\frac{T_s^2}{2}\\ T_s\end{bmatrix}\overline{u}(k) + \boldsymbol{w}(k) \tag{23}$$

with state $\boldsymbol{x}(k)\in\mathbb{R}^2$, input set $\mathcal{U} = [-1,1]$, Gaussian disturbance $\boldsymbol{w}(k)$ with mean $\overline{\mu}_{\boldsymbol{w}}(k) = [0\ 0]^\top$ and covariance matrix $C_{\boldsymbol{w}}(k) = 10^{-4}I_2$, sampling time $T_s = 0.25$s, and initial position $x(0)\sim\mathcal{N}([0.4\ 0]^\top, C_{\boldsymbol{x}})$. For a time horizon of $N\in\mathbb{N}$, the polytopic safe set $\mathscr{S}$ is obtained by projecting $\mathscr{T}$, a time-varying target tube that imposes constraints on the position, to $\mathbb{R}^{nN+1}$, to generate constraints on $\boldsymbol{X}_{\overline{U}}$,

$$\mathscr{T} = \left\{(t,\overline{x})\in\mathbb{N}_{[0,N]}\times\mathbb{R}^2 : m_1 t + c_1 \leq \overline{x}_1 \leq m_2 t + c_2\right\}.$$

We use a quadratic cost (7) to track $\overline{X}_{\mathrm{d}}\in\mathbb{R}^{nN+1}$, penalize high velocities, and minimize control effort. We choose $Q = \mathrm{diag}([10\ \ 1])\otimes I_{(N+1)\times(N+1)}$, $R = 10^{-3}I_{N\times N}$, $(\overline{X}_{\mathrm{d}})_t = [m_r t + c_r\ \ 0]^\top$, $\forall t\in\mathbb{N}_{[0,N]}$, and set $m_1, m_2, m_r, c_1, c_2, c_r$ as $0.15, -0.15, 0.3, -0.5, 0.5, -0.4$ respectively.

*1) Fixed time horizon ($N = 10$ time steps):* We analyze the stochastic motion planning problem (6) with safety probability threshold $1-\Delta\in\{0.4, 0.8\}$ and $C_{\boldsymbol{x}}\in$

| Example | Double integrator example (Section V-A, Figure 3) | | | | | | Spacecraft rendezvous example (Section V-B, Figure 5) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Safety probability threshold | $1-\Delta = 0.8$ | | | | $1-\Delta = 0.4$ | | $1-\Delta = 0.8$ | | | | $1-\Delta = 0.4$ | |
| Approach | QP (22) | IRA [10] | MIQP (17) | PC [1] | MIQP (17) | PC [1] | QP (22) | IRA [10] | MIQP (17) | PC [1] | MIQP (17) | PC [1] |
| Solve time (s) | 0.05 | 0.38 | 0.39 | 13.64 | 0.83 | 10.59 | 0.06 | 1.32 | 0.23 | 14.13 | 2.09 | 86.19 |
| Relative absolute error between the simulated and expected costs ($\times 10^{-3}$) | 9.88 | 9.95 | 10.11 | 30.31 | 94.43 | 64.98 | 5.85 | 5.73 | 5.93 | 4.19 | 6.18 | 7.74 |
| Safety probability | 0.85 | 0.85 | 0.84 | 0.84 | 0.71 | 0.60 | 0.81 | 0.80 | 0.83 | 0.78 | 0.58 | 0.39 |

TABLE I: Comparison of solutions to (9). We propose a quadratic program (QP) to solve (22) for $1-\Delta \geq 0.5$ and a mixed-integer quadratic program (MIQP) to solve (17), and compare them to iterative risk allocation (IRA) [10] and particle control (PC) [1] (100 particles), respectively. A Monte-Carlo simulation ($10^5$ particles) validates performance and safety probability obtained using each controller.

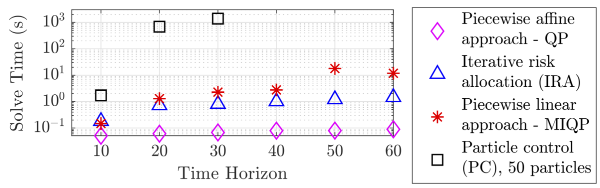


Fig. 4. Solve times for various approaches in the double integrator example

$\{10^{-3}I_2, 10^{-4}I_2\}$ respectively. Figure 3 shows that trajectories produced by all the methods are very similar. Table I tabulates the computation time and Monte-Carlo simulation-based validation of the optimal controllers. All methods reproduce acceptable relative absolute error between the simulated and the expected cost, and the simulated safety probability under Monte-Carlo simulation. The solve time of our algorithms are faster than their counterparts: the QP in (22) is faster than IRA, and the solve time for MIQP (17) is faster than PC. We used 100 particles for the PC approach.

*2) Different time horizons $N \in \mathbb{N}_{[10,60]}$:* We next analyze how the computation time scales with time horizon $N$ for the double integrator example with safety probability threshold $1-\Delta = 0.8$ and $C_{\boldsymbol{x}} = 10^{-4}I_2$. Figure 4 shows that, due to the non-iterative nature, the QP scales significantly better than the IRA [10]. Since PC [1] and MIQP are mixed-integer formulations, their solve time increases exponentially with $N$. However, our MIQP scales better, potentially due to the structure afforded by (17). For reasonable computation times, we used only 50 particles for the PC approach.

### B. Spacecraft rendezvous example

We consider two spacecraft in the same elliptical orbit. One spacecraft, referred to as the deputy, must approach and dock with another spacecraft, referred to as the chief, while remaining in a line-of-sight cone, in which accurate sensing of the other vehicle is possible. The relative dynamics are described by the Clohessy-Wiltshire-Hill (CWH) equations [24] with additive stochastic noise,

$$\ddot{x} - 3\omega x - 2\omega\dot{y} = m_d^{-1}F_x, \qquad \ddot{y} + 2\omega\dot{x} = m_d^{-1}F_y. \quad (24)$$

The chief is located at the origin, the position of the deputy is $x, y \in \mathbb{R}$, $\omega = \sqrt{\mu/R_0^3}$ is the orbital frequency, $\mu$ is the gravitational constant, and $R_0$ is the orbital radius of the spacecraft. See [2] for further details and numerical values.

We define the state as $z = [x, y, \dot{x}, \dot{y}] \in \mathbb{R}^4$ and the input as $u = [F_x, F_y] \in \mathcal{U} \subset \mathbb{R}^2$. We discretize (24) to

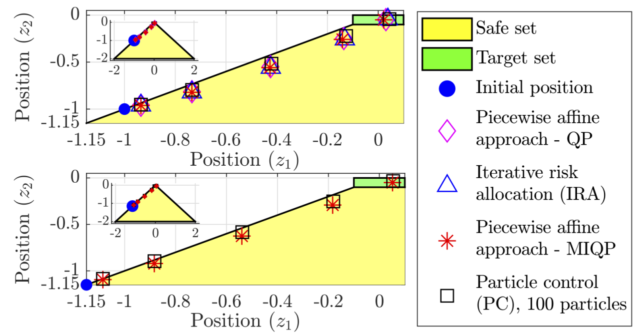


Fig. 5. Spacecraft rendezvous problem with safety probability threshold $1-\Delta = 0.8$ (top plot) and $1-\Delta = 0.4$ (bottom plot). The insets display the trajectory with the entire safe set.

obtain a LTI system $\boldsymbol{z}(k+1) = A\boldsymbol{z}_k + B\overline{u}(k) + \boldsymbol{w}(k)$ with input space $\mathcal{U} = [-0.1, 0.1]^2$, and Gaussian disturbance $\boldsymbol{w}(k) \in \mathbb{R}^4$ with mean $\overline{\mu}_{\boldsymbol{w}}(k) = [0\ 0]^\top$ and covariance matrix $C_{\boldsymbol{w}}(k) = 10^{-4} \times \mathrm{diag}(1, 1, 5\times 10^{-4}, 5\times 10^{-4})$. We define the target set and the constraint set as in [2]

$$\begin{aligned}\mathcal{T} = \{\overline{z} \in \mathbb{R}^4 : &|\overline{z}_1| \leq 0.1, -0.1 \leq \overline{z}_2 \leq 0,\\ &|\overline{z}_3| \leq 0.01, |\overline{z}_4| \leq 0.01\} \quad (25)\end{aligned}$$

$$\mathcal{K} = \{\overline{z} \in \mathbb{R}^4 : |\overline{z}_1| \leq \overline{z}_2, |\overline{z}_3| \leq 0.05, |\overline{z}_4| \leq 0.05\} \quad (26)$$

with a horizon of $N = 5$. Thus, $\mathscr{S} = \mathcal{K}^4 \times \mathcal{T}$. To drive the spacecraft to origin, we use a quadratic cost (7) with $Q = \mathrm{diag}([10\ 1\ 10\ 1]) \otimes I_{6\times 6}$, $R = 10^{-3}I_{10\times 10}$, and $\forall t \in \mathbb{N}_{[0,N]}, \overline{X}_{\mathrm{d}} = [0\ 0\ 0\ 0]^\top$. We presume a safety probability threshold $1-\Delta \in \{0.4, 0.8\}$ and deterministic initial states $\mu_{\boldsymbol{x}} \in \{[-1\ \ -1\ 0\ 0]^\top, [-1.15\ \ -1.15\ 0\ 0]^\top\}$ ($C_{\boldsymbol{x}} = 0$).

Figure 5 shows the optimal trajectory taken by each of the methods. Table I tabulates the computation time and Monte-Carlo simulation-based validation of the optimal controllers. Similarly to Section V-A, the solve time of our algorithms is faster than their counterparts: the QP in (22) is faster than IRA, and the solve time for MIQP (17) is faster than PC. Note that the simulated relative absolute error between the simulated and the expected cost, and simulated safety probability, are within acceptable ranges. We used 100 particles for the PC approach.

## VI. Conclusion and Future Work

This paper utilizes risk allocation and piecewise affine approximations in two conservative solutions to a stochastic optimal control problem with Gaussian-perturbed linear dynamics, soft polytopic state constraints, hard polytopic input constraints, and a convex cost function. When the safety probability threshold is above 0.5, we propose a convex program to solve the optimal control problem in a non-iterative manner. For the general problem, we propose a mixed-integer convex program that is free from sampling. For a quadratic cost function, these approaches simplify to a quadratic and mixed-integer quadratic program, respectively. Using two examples, we demonstrate that the proposed approaches outperform existing iterative risk allocation [10] and the particle control [1] approaches in computation time, without compromising on the solution quality.

## APPENDIX

### A. Piecewise-affine approximations for concave functions

We show that, given a concave function $f : \mathcal{D} \to \mathcal{R}$ where $\mathcal{D}, \mathcal{R}$ are bounded intervals in $\mathbb{R}$ and $f$ has a well-defined gradient and hessian, Algorithm 1 computes its PWA under- and overapproximation (11) that satisfies (12).

**Algorithm 1** Piecewise-affine (PWA) approximations $\ell_f^+, \ell_f^-$

**Input:** Concave $f : \mathcal{D} \to \mathcal{R}$ with gradient $\nabla f(x)$ and hessian $\nabla^2 f(x)$, bounded sets $\mathcal{D} = [x_{\min}, x_{\max}], \mathcal{R} \subset \mathbb{R}$, maximum approximation error $\eta > 0$

**Output:** PWA over- ( $\ell_f^+$) and underapproximation ($\ell_f^-$)

1: $x_j \leftarrow x_{\min}$ and $j \leftarrow 1$
2: **while** $x_j < x_{\max}$ **do**
3: Solve $h^2 \left( \min_{x \in [x_j, x_j + h]} \nabla^2 f(x) \right) + 8\eta = 0$ for $h$
4: $h \leftarrow \min(h, x_{\max} - x_j),\ x_{j+1} \leftarrow x_j + h$
5: *//Construction of PWA underapproximation ($\ell_f^-$)*
6: $m_{f,j}^- \leftarrow \frac{f(x_{j+1}) - f(x_j)}{h},\ c_{f,j}^- \leftarrow \frac{f(x_{j+1}) x_j - (x_{j+1}) f(x_j)}{h}$
7: *//Construction of PWA overapproximation ($\ell_f^+$)*
8: Solve $\nabla f(y_j) = m_{f,j}^-$ for $y_j \in [x_j, x_{j+1}]$
9: $m_{f,j}^+ \leftarrow \nabla f(y_j),\ c_{f,j}^+ \leftarrow f(y_j) - \nabla f(y_j) y_j$
10: Increment $j$ by 1
11: **end while**
12: $N_f \leftarrow j$ and $N_f \leftarrow j - 1$
13: $\ell_f^+(y) \leftarrow \min_{j \in \mathbb{N}_{[1, N_f]}} \{ m_{f,j}^+ x + c_{f,j}^+ \}$
14: $\ell_f^-(y) \leftarrow \min_{j \in \mathbb{N}_{[1, N_f]}} \{ m_{f,j}^- x + c_{f,j}^- \}$

Recall that for linear Lagrange interpolation, the approximation error in the interval $[x_j, x_j + h]$ for any $x_j \in \mathcal{D}$ and $h > 0$ is bounded from above by $\frac{h^2 \min_{x \in [x_j, x_j + h]} \nabla^2 f(x)}{8}$ [20, Eq. 25.2.1]. Since $f(x)$ is concave, $\nabla^2 f(x)$ is non-positive, and the line joining $(x_j, f(x_j)$ and $(x_j + h, f(x_j + h))$ underapproximates $f(x)$ in the interval $[x_j, x_j + h]$ [19, Sec. 3.1]. Thus, line 3 of Algorithm 1 computes the interval gap $h$ by ensuring that the underapproximation error is below $\eta$. However, solving line 3 is hard without imposing additional structure on $f(x)$. For $f(x)$ with monotone $\nabla^2 f(x)$ in the interval $[x_j, x_j + h]$,

$$\min_{x \in [x_j, x_j + h]} \nabla^2 f(x) = \begin{cases} \nabla^2 f(x_j), & \text{nondecreasing } \nabla^2 f(x) \\ \nabla^2 f(x_j + h), & \text{nonincreasing } \nabla^2 f(x) \end{cases}$$

For these cases, line 3 simplifies to a problem of finding the root of the equation $h^2(\nabla^2 f(x_j)) + 8\eta = 0$ or $h^2(\nabla^2 f(x_j + h)) + 8\eta = 0$ respectively. The nonlinear functions of interest in this paper, $\Phi^{-1}(1 - x)$, $\log \Phi(x)$, and $\log(1 - x)$, have monotone second derivatives, which permit an efficient implementation of Algorithm 1.

**Remark 3.** *Algorithm 1 can be easily adapted for convex functions as negations of convex functions are concave.*

## REFERENCES


[1] L. Blackmore, M. Ono, and B. Williams, "Chance-constrained optimal path planning with obstacles," *IEEE Trans. Robot.*, vol. 27, no. 6, pp. 1080–1094, 2011.

[2] K. Lesser, M. Oishi, and R. S. Erwin, "Stochastic reachability for control of spacecraft relative motion," in *Proc. IEEE Conf. Dec. & Ctrl.*, 2013, pp. 4705–4712.

[3] B. HomChaudhuri, A. Vinod, and M. Oishi, "Computation of forward stochastic reach sets: Application to stochastic, dynamic obstacle avoidance," in *Proc. Amer. Ctrl. Conf.*, 2017, pp. 4404–4411.

[4] M. Vitus, Z. Zhou, and C. Tomlin, "Stochastic control with uncertain parameters via chance constrained control," *IEEE Trans. Autom. Ctrl.*, vol. 61, no. 10, pp. 2892–2905, 2016.

[5] D. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[6] M. Farina, L. Giulioni, and R. Scattolini, "Stochastic linear model predictive control with chance constraints–a review," *J. Process Ctrl.*, vol. 44, pp. 53–67, 2016.

[7] A. Mesbah, "Stochastic model predictive control: An overview and perspectives for future research," *IEEE Ctrl. Syst. Mag.*, vol. 36, no. 6, pp. 30–44, 2016.

[8] G. Calafiore and M. Campi, "The scenario approach to robust control design," *IEEE Trans. Autom. Ctrl.*, vol. 51, no. 5, pp. 742–753, 2006.

[9] F. Oldewurtel, C. Jones, A. Parisio, and M. Morari, "Stochastic model predictive control for building climate control," *IEEE Trans. Control Syst. Technol.*, vol. 22, no. 3, pp. 1198–1205, 2014.

[10] M. Ono and B. Williams, "Iterative risk allocation: A new approach to robust model predictive control with a joint chance constraint," in *Proc. IEEE Conf. Dec. & Ctrl.*, 2008, pp. 3427–3432.

[11] M. Vitus and C. Tomlin, "On feedback design and risk allocation in chance constrained control," in *Proc. IEEE Conf. Dec. & Ctrl.*, 2011, pp. 734–739.

[12] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming," http://cvxr.com/cvx, Mar. 2014.

[13] Gurobi Optimization LLC, "Gurobi optimizer reference manual," 2018. [Online]. Available: http://www.gurobi.com

[14] A. Nemirovski and A. Shapiro, "Convex approximations of chance constrained programs," *J. Optimization*, vol. 17, pp. 969–996, 2006.

[15] A. Shapiro, D. Dentcheva, and A. Ruszczyński, *Lectures on stochastic programming: modeling and theory*. SIAM, 2009.

[16] R. Horst, P. Pardalos, and N. Van Thoai, *Introduction to Global Optimization*. Springer US, 2000.

[17] J. Skaf and S. Boyd, "Design of affine controllers via convex optimization," *IEEE Trans. Autom. Ctrl.*, vol. 55, pp. 2476–2487, 2010.

[18] P. Kumar and P. Varaiya, *Stochastic systems: Estimation, identification, and adaptive control*. SIAM, 1986, vol. 75.

[19] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge Univ. Press, 2004.

[20] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, & Mathematical Tables*. Courier Corp., 1965.

[21] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, pp. 407–427, 1999.

[22] M. Herceg, M. Kvasnica, C. Jones, and M. Morari, "Multi-Parametric Toolbox 3.0," in *Proc. European Ctrl. Conf.*, July 2013, pp. 502–510, http://control.ee.ethz.ch/∼mpt.

[23] A. Vinod, J. Gleason, and M. Oishi, "SReachTools: Stochastic reachability toolbox for MATLAB," in *Proc. Hybrid Syst.: Comput. and Ctrl.*, 2019, https://unm-hscl.github.io/SReachTools (accepted).

[24] W. E. Weisel, *Spaceflight dynamics*. New York, McGraw-Hill Book Co, 1989, vol. 2.